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Classical Analog of Quantum Phase

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A modified version of the Feynman relativistic chessboard model (FCM) is investigated in which the paths involved are spirals in space-time. Portions of the paths in which the particle's proper time is reversed are interpreted in terms of antiparticles. With this interpretation the particle–antiparticle field produced by such trajectories provides a classical analog of the phase associated with particle paths in the unmodified FCM. It is shown that in the nonrelativistic limit the resulting kernel is the correct Dirac propagator and that particle–antiparticle symmetry is in this case responsible for quantum interference.

1. INTRODUCTION

One of the appealing features of the path integral formulation of nonrelativistic quantum mechanics (Feynman and Hibbs, 1965) is the explicit use of classical paths. In the expression

$$\mathbf{K}(b, a) = \sum_{[x(t)]} e^{iS([x(t)])} \tag{1.1}$$

where the sum is over all classical paths $[x(t)]$ between a and b, it is only the association of phase with paths which marks the kernel as a nonclassical object. Here the operative "phase rule" is to associate the classical action S of the path $[x(t)]$ in units of $\hbar$ with the phase angle of the path. Such a phase rule is not unique, and is provided as an ansatz for the particular model in question.

Expressions of the form (1.1) provide a convenient illustration of the general extent of the quantum–classical analogy. The paths themselves and their use in a propagator have common analogs in classical systems. However, the association of a phase with each classical path is apparently outside the domain of classical physics. It is, however, this association which is at

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the heart of the phenomenon of quantum interference, and gives rise to many of the puzzling aspects of quantum theory.

To say that there is no classical analog of the association of phase with path is, however, too strong a statement. In the 1940s Feynman and Wheeler, and Stuckelberg (1948) pointed out that there is a classical analog of the antiparticle. For example, Feynman (1948) considered finding the paths of extreme action of a model electron encountering a high potential barrier (Figure 1). He noted that under certain circumstances, paths with time-reversed sections provided such extrema, and the time-reversed segments could be interpreted as positrons. This interpretation provided charge conservation in time. However, viewing the trajectory as a continuous space-time curve, it also provides the charge on the electron with a classical "phase" which changes by a factor of π with each reversal of the direction in time. For future reference we shall call this the classical phase rule.

Since this rule has a classical basis in a least action principle, it seems reasonable to investigate its relationship to the quantum mechanical phase associated with paths in nonrelativistic quantum mechanics.

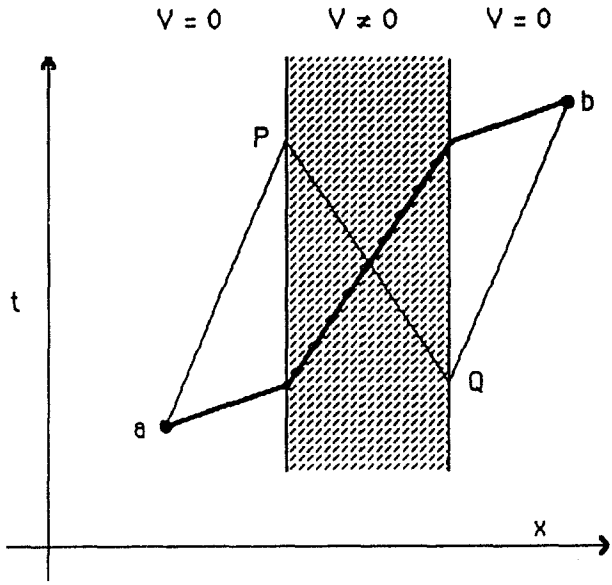

Fig. 1. If two points a and b are separated by a high potential barrier, there may be two paths which make the classical action an extremum. One (heavy line) represents the passage of a fast electron which is slowed down by the barrier. The other has a section which is reversed in time and is interpreted as the penetration of the barrier by a slow electron by means of a pair production at Q and an annihilation at P, section PQ representing the motion of the positron (Feynman, 1948).

It may be noted that, at the outset, it is not clear that there is any relationship whatsoever between the two concepts. The phase associated with equation (1.1) is a purely quantum mechanical object with no direct connection to special relativity (or indeed to anything in classical physics). On the other hand, the classical phase rule is a consequence of special relativity and has an independent existence within classical physics. It appears that the two concepts live in completely separate worlds, although the history of their association goes back a long way.[2]

In this paper we begin an investigation by considering the specific case of a free particle moving in one dimension. The model we use is a modification of the Feynman chessboard model (FCM).

The FCM has a very simple phase rule which produces the correct Dirac propagator in the nonrelativistic limit. Although the phase rule is simple, it is without any obvious classical motivation. The modification of the FCM which we propose involves paths which are spirals in space-time. In the modified model the phase rules invoked are directly related to the classical phase rule associated with reversed proper time. The two phase rules considered yield propagators for the Dirac and Klein–Gordon equations, respectively. In both cases we find that it is the simple requirement of charge conservation in time that generates the oscillatory character of the quantum propagators. Furthermore, each individual path in the sum over paths produces a field of finite extent in space, providing a direct analog of wave–particle duality. Thus, the formulation provides a rather unique "picture" of a particle moving in one dimension, and we hope that it will encourage further investigation into more realistic systems.

In Section 2 we review the FCM using both the 2×2 matrix formulation of Gersch (1981) and Jacobson and Schulman (1984) and the 4×4 formulation of Ord (1992). In Section 3 we modify the 4×4 formulation to incorporate spiral trajectories in space-time, and in Section 4 we discuss the results.

2. TRANSFER MATRIX FORMULATION

In the FCM a point particle is constrained to move with speed $c = 1$ on a space-time lattice with lattice spacing ε. The kernel for a particle to propagate from position a at time t_a to position b at time t_b is given by

$$\mathbf{K}(b, a) = \sum_R N(R)(i\varepsilon m)^R \tag{2.1}$$

where the sum is over all forward "bishop's moves" connecting the space-time points (Figure 2). The set of paths is subdivided into paths with R

[2]For a very interesting article containing some of the history of this see Schweber (1986).

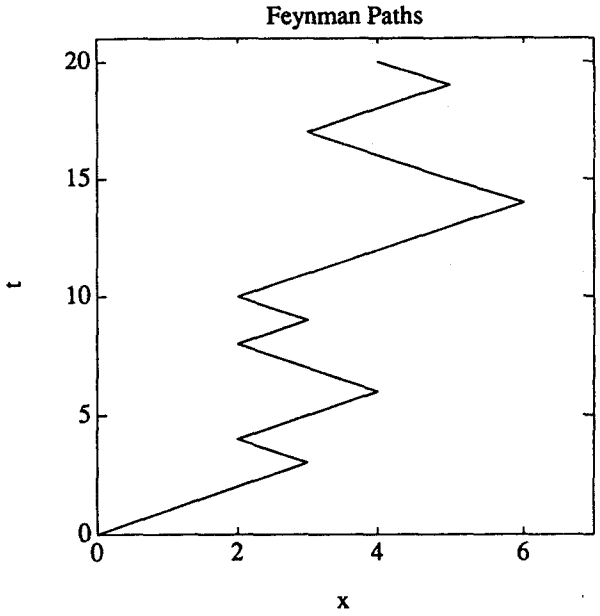

Fig. 2. A path for the Feynman chessboard model.

corners, and any path with R corners is given a weight $(i\varepsilon m)^R$. $N(R)$ is the number of paths with R corners. In the limit as $\varepsilon \to 0$, provided $(t_b - t_a) \gg 1/m$, the above free-particle propagator approaches the correct Dirac free-particle kernel (Feynman and Hibbs, 1965).

The combinatorial factors $N(R)$ in (2.1) may be calculated explicitly. However, a more instructive approach is to consider a spin formulation.

Referring to Figure 2, we can see that the Feynman paths are constructed from only two kinds of elementary links. These links may be labeled + or − according to their displacement in the x direction. Thus at any lattice point on the t axis, t_i say, the displacement of the path $X(t_i) = X_i$ is simply $X_i = X_{i-1} + \varepsilon\sigma_i$, where ε is the lattice spacing and $\sigma_i = \pm 1$ is an Ising spin variable. This suggests that we may represent any N-step Feynman path by a set of N spins $\{\sigma_1, \ldots, \sigma_N\}$.

The sum over paths, equation (2.1) (which is actually a 2×2 matrix with elements indexed by the orientation of the first and last link), is then

$$\mathbf{Z}_{\sigma_N,\sigma_1}(M, N, \varepsilon) = \sum_R N(R)(i\varepsilon m)^R$$

$$= \left(\sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \right)' (i\varepsilon m)^{\sum_{i=1}^{N-1} (1 - \sigma_i \sigma_{i+1})}. \qquad (2.2)$$

Here we have chosen to make the ε and N dependence explicit and we have labeled the spatial displacement by the integer $M=(b-a)/\varepsilon$. In the above, the sum in the exponent counts the number of corners in the path, and the prime on the sums indicates that only configurations such that $\sum_{i=1}^{N-1}\sigma_i=M$ are considered. Note that apart from the presence of i in the weighting factor $(i\varepsilon m)$, the quantity $\mathbf{Z}$ is formally a partition function for a one-dimensional Ising chain.

The sums in (2.2) are awkward to calculate because of the restriction to constant displacement M. However, if we consider the discrete Fourier transform of $\mathbf{Z}$, i.e.,

$$\mathbf{Z}_{\sigma_N,\sigma_1}(p, N, \varepsilon)=\sum_{M=-N}^{N} e^{-iM\varepsilon p}\mathbf{Z}_{\sigma_N,\sigma_1}(M, N, \varepsilon) \tag{2.3}$$

this removes the constraint of fixed M and is easier to calculate. We may recover the fixed displacement form (2.2) by considering an integral form of the Kronecker delta, namely

$$\delta_{MM'}=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{i(M-M')\phi} \tag{2.4}$$

Then from (2.3)

$$\mathbf{Z}_{\sigma_N,\sigma_1}(M', N, \varepsilon)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d(p\varepsilon)\, \mathbf{Z}_{\sigma_N,\sigma_1}(p, N, \varepsilon)\, e^{iM'p\varepsilon} \tag{2.5}$$

Now to calculate (2.3) we write

$$j=-\tfrac{1}{2}\ln(i\varepsilon m) \tag{2.6}$$

and using (2.2), we have

$$\mathbf{Z}_{\sigma_N,\sigma_1}(p, N, \varepsilon)=\sum_{\sigma_2=\pm 1}\cdots\sum_{\sigma_{N-1}=\pm 1}\exp\left[-ip\varepsilon\sum_{n=1}^{N}\sigma_n-j\sum_{n=1}^{N-1}(1-\sigma_n\sigma_{n+1})\right] \tag{2.7}$$

Here the sums are unconstrained and (2.7) may be evaluated using a transfer matrix. That is, writing

$$\mathbf{T}_F(\sigma, \sigma')=\exp[-(i/2)p\varepsilon(\sigma+\sigma')+ij(\sigma\sigma'-1)] \tag{2.8}$$

we find that equation (2.7) becomes

$$\mathbf{Z}_{\sigma_N,\sigma_1}(p, N, \varepsilon) = e^{-ip\varepsilon(\sigma_1+\sigma_N)}(\mathbf{T}_F^{N-1})_{\sigma_N,\sigma_1} \tag{2.9}$$

Now we are interested in the limit of $\mathbf{Z}$ as $\varepsilon \to 0$ in such a way that $t = N\varepsilon$ is fixed. We thus have to find

$$\mathbf{K}(p, t) = \lim_{\varepsilon \to 0} \mathbf{T}_F^{t/\varepsilon} \tag{2.10}$$

To lowest order in ε, $\mathbf{T}_F$ is

$$\mathbf{T}_F = \begin{pmatrix} 1 - ip\varepsilon & i\varepsilon m \\ i\varepsilon m & 1 + ip\varepsilon \end{pmatrix} \tag{2.11}$$

The eigenvalues of $\mathbf{T}_F$ are

$$\lambda_\pm^F = 1 \pm i\varepsilon E \tag{2.12}$$

with

$$E = (m^2 + p^2)^{1/2} \tag{2.13}$$

The orthogonal projection operators corresponding to $\mathbf{T}_F$ are

$$\mathbf{P}_F^\pm = \frac{1}{2}\left(\mathbf{1} \pm \frac{m\boldsymbol{\sigma}_x - p\boldsymbol{\sigma}_z}{E}\right) \tag{2.14}$$

where $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_z$ are Pauli spin matrices and $\mathbf{1}$ is the 2×2 identity. In the limit as $\varepsilon \to 0$, $\lambda_\pm^{t/\varepsilon} \to e^{\pm iEt}$ and (2.8) becomes

$$\mathbf{K}(p, t) = \sum_{\mu = \pm 1} \frac{1}{2}\left[\mathbf{1} + \frac{\mu}{E}(m\boldsymbol{\sigma}_x - p\boldsymbol{\sigma}_z)\right] e^{i\mu Et} \tag{2.15}$$

This propagator satisfies the Dirac equation (Jacobson and Schulman, 1984) and is the Fourier transform of the kernel (2.1) in the limit of small lattice spacing.

The above calculation of (2.15) relies on the fact that the Feynman paths in this case all have the same displacements in t (i.e., $N\varepsilon$). In a subsequent calculation this will not be the case, so for future reference we will incorporate paths of all lengths. To do this, we simply modify the transfer matrix (2.8) so that the displacement in time is "counted" by a variable $q\varepsilon$. Since all transitions in the Feynman paths advance one unit in time per step, the new transfer matrix to lowest order in ε is

$$\begin{aligned} \mathbf{T}_F'(\boldsymbol{\sigma}, \boldsymbol{\sigma}') &= \exp[-(i/2)p\varepsilon(\sigma + \sigma') + ij(\sigma\sigma' - 1)] \exp(-iq\varepsilon) \\ &= \begin{pmatrix} 1 - i\varepsilon(p+q) & \varepsilon m \\ \varepsilon m & 1 + i\varepsilon(p-q) \end{pmatrix} \end{aligned} \tag{2.16}$$

In order to find $\mathbf{K}(p, t)$ in this case, we have to sum over paths of arbitrary length and extract only those with displacement t in time. Thus,

$$\mathbf{K}(p, t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} d(q\varepsilon) \sum_{n=0}^{\infty} (\mathbf{T}_{\mathrm{F}}')^n e^{iq\varepsilon N} \tag{2.17}$$

where the limit is such that $t = \varepsilon N$ is fixed. Comparing this with (2.5), we see that q is conjugate to t in the same way that p is conjugate to the walk displacement $X = \varepsilon M$. Expanding $\mathbf{T}_{\mathrm{F}}'$ to the lowest order in ε, we find, using (2.13) and (2.14), that

$$\mathbf{T}_{\mathrm{F}}' = [1 + i\varepsilon(E - q)]\mathbf{P}_{+}^{\mathrm{F}} + [1 - i\varepsilon(E + q)]\mathbf{P}_{-}^{\mathrm{F}} \tag{2.18}$$

where $\mathbf{P}_{\pm}^{\mathrm{F}}$ are the orthogonal projection operators for $\mathbf{T}_{\mathrm{F}}$. Now summing the series in (2.17), we have

$$\begin{aligned} \mathbf{K}(p, t) &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} dq \left(\frac{\mathbf{P}_{+}^{\mathrm{F}}}{q - E} + \frac{\mathbf{P}_{-}^{\mathrm{F}}}{q + E} \right) e^{iqt} \\ &= \mathbf{P}_{+}^{\mathrm{F}} e^{iEt} + \mathbf{P}_{-}^{\mathrm{F}} e^{-iEt}, \qquad t > 0 \end{aligned} \tag{2.19}$$

Here we evaluated the integrals by moving the poles at $q = \pm E$ slightly into the positive half-plane and completing the contour as a semicircle in the upper half-plane. This gives the retarded propagator, with $\mathbf{K}(p, t) = 0$ for $t < 0$, and reproduces the result (2.15).

The above calculations of the transformed kernel (2.19) are fairly simple and direct; however, the role of the phase rule implicit in (2.1) is not entirely transparent. In terms of the transfer matrices $\mathbf{T}_F$ and $\mathbf{T}_F'$ of (2.8) and (2.16) the path phases enter through the imaginary off-diagonal elements $(i\varepsilon m)$. In terms of a spin system this corresponds to an Ising chain with imaginary Boltzmann weights. Since we wish to separate, where possible, the phase rule from the path statistics, it is worthwhile reformulating the problem so that this is done in an obvious way.

We note that the sum in equation (2.1) may be rewritten for finite ε as

$$\begin{aligned} \mathbf{K}(b, a) = &\sum_{R=0,4,\ldots} N(R)(\varepsilon m)^R - \sum_{R=2,6,\ldots} N(R)(\varepsilon m)^R \\ &+ i\left(\sum_{R=1,5,\ldots} N(R)(\varepsilon m)^R - \sum_{R=3,7,\ldots} N(R)(\varepsilon m)^R \right) \end{aligned} \tag{2.20}$$

Here the periodicity of i^R is used to expand the sum into four separate terms. Each individual sum is a classical weighted sum pertaining to a restricted class of walks on the space-time lattice. The two minus signs and the presence of i in (2.20) combine the classical sums to form a quantum propagator. To segregate the phase rule from the path statistics we shall calculate the Fourier

transforms of each of the sums in (2.20) separately. To do this, we consider the transfer matrix

$$\mathbf{T}_4 = \begin{pmatrix} e^{-ip\varepsilon} & 0 & 0 & \varepsilon m \\ 0 & e^{-ip\varepsilon} & \varepsilon m & 0 \\ \varepsilon m & 0 & e^{ip\varepsilon} & 0 \\ 0 & \varepsilon m & 0 & e^{ip\varepsilon} \end{pmatrix} \tag{2.21}$$

Here states one to four correspond to contributions to the respective sums in equation (2.20). States one and two correspond to steps in the $+x$ direction, and are distinguished by the parity of the number of corners in their "history." States three and four correspond to steps in the $-x$ direction.

As was discussed elsewhere (Ord, 1992), the simple structure of (2.21) is easily interpreted. The diagonal elements of $\mathbf{T}_4$ correspond to two steps in the same direction. Such successive steps either increase or decrease the "displacement counting" variable $p\varepsilon$ through multiplication by $e^{\pm ip\varepsilon}$. Two adjacent steps in opposite directions do not change the net displacement, but do contribute a real, positive corner weight εm. The allowed state changes from the structure of (2.20) are $1 \rightarrow 3$, $3 \rightarrow 2$, $2 \rightarrow 4$, and $4 \rightarrow 1$. This is reflected in the respective entries in $\mathbf{T}_4$.

Now $\mathbf{T}_4^{N-1}$ simply counts the N-step walks involved in (2.20) while maintaining a distinction between walks with different numbers of corners modulo 4. To reconstruct the finite-ε propagator of equation (2.9), we have to contract to a 2×2 representation using the prescription provided by (2.20). That is, we have to implement the phase rule (subsequently called the Feynman phase rule) which associate a phase of $\pi/2$ for every corner in the path. To do this, we regard states one and three as being the two distinct states of the chessboard problem and we "start" walks in these states. Walks which end in states two or four contribute -1 times their statistical weight and are thus subtracted from the contributions to states one and three, respectively. Finally, the factor of i in (2.20) distinguishes the two states of the chessboard problem.

To perform the operation of adding the separate sums of (2.20) with the appropriate signs as above, we consider the "contractors"

$$\mathbf{q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}^T \tag{2.22}$$

and

$$\mathbf{q}^* = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & i & -i \end{pmatrix} \tag{2.23}$$

To reproduce the kernel (2.9), we then have

$$\mathbf{K}(p, t) = \mathbf{q}^* \left(\lim_{\varepsilon \to 0} \mathbf{T}_4^N \right) \mathbf{q} \tag{2.24}$$

The advantage of the above formulation is that the expression in brackets above is completely classical and easily interpreted in terms of lattice walks. The "quantum mechanics" is brought in through the contractors $\mathbf{q}$ and $\mathbf{q}^*$ which implement the phase rule of the model.

To verify that (2.24) is in fact correct, we first note that to lowest order in ε,

$$\mathbf{q}^* \mathbf{T}_4 \mathbf{q} = \begin{pmatrix} e^{-ip\varepsilon} & i\varepsilon m \\ i\varepsilon m & e^{ip\varepsilon} \end{pmatrix} = \mathbf{T}_F \tag{2.25}$$

Thus, $\mathbf{T}_4$ contracts to the transfer matrix of the 2×2 representation (2.8). However, we need more than this to confirm (2.24). We write $\mathbf{T}_4$ in terms of its eigenvalues and orthogonal projection operators. For small ε these are

$$\mathbf{T}_4 = \sum_{\mu = \pm} (\lambda_\mu^{\mathrm{c}} \mathbf{P}_\mu^{\mathrm{c}} + \lambda_\mu^{\mathrm{R}} \mathbf{P}_\mu^{\mathrm{R}}) \tag{2.26}$$

with

$$\lambda_\pm^{\mathrm{c}} = 1 \pm i\varepsilon E \tag{2.27}$$

and

$$\lambda_\pm^{\mathrm{R}} = 1 \pm \varepsilon (m^2 - p^2)^{1/2} = 1 \pm \varepsilon F \tag{2.28}$$

The projectors are

$$\mathbf{P}_{\pm \cdot}^{\mathrm{c}} = \begin{pmatrix} (1 \mp p/E)\Sigma^- & (im/E)\Sigma^- \\ -(im/E)\Sigma^- & (1 \pm p/E)\Sigma^- \end{pmatrix} \tag{2.29}$$

and

$$\mathbf{P}_{\pm \cdot}^{\mathrm{R}} = \begin{pmatrix} (1 \mp p/F)\Sigma^+ & (m/F)\Sigma^+ \\ (m/F)\Sigma^+ & (1 \pm p/F)\Sigma^+ \end{pmatrix} \tag{2.30}$$

with

$$\Sigma^\pm = \tfrac{1}{4}(\mathbf{1} \pm \boldsymbol{\sigma}_x) \tag{2.31}$$

Comparing the complex eigenvalues $\lambda_\pm^{\mathrm{c}}$ of (2.27) with the eigenvalues of the 2×2 transfer matrix (2.12) and (2.13), we see that they are identical. It can also be verified that the real eigenvalues $\lambda_\pm^{\mathrm{R}}$ correspond to the associated

Ising partition function [i.e., equation (2.1) with real corner weights εm]. Furthermore, it is easily verified that

$$\mathbf{q}^*\mathbf{P}^{\mathrm{c}}_{\pm}\mathbf{q}=\mathbf{P}^{\mathrm{F}}_{\pm} \tag{2.32}$$

and

$$\mathbf{q}^*\mathbf{P}^{\mathrm{R}}_{\pm}\mathbf{q}=0 \tag{2.33}$$

Thus, the contractors $\mathbf{q}^*$ and $\mathbf{q}$ select only projections of $\mathbf{T}_4^N$ onto the eigenspace corresponding to the two complex eigenvalues. Similarly, it may be shown that the contractors reduce the orthogonal projectors of $\mathbf{T}_4$ corresponding to the complex eigenvalues $\lambda^{\mathrm{c}}_{\pm}$ to their respective projectors in the 2×2 representation. Thus, from (2.26), (2.32), and (2.33)

$$\mathbf{q}^*\mathbf{T}_4^N\mathbf{q}=\mathbf{q}^*\sum_{\mu=\pm}((\lambda^{\mathrm{c}}_{\mu})^N\mathbf{P}^{\mathrm{c}}_{\mu}+(\lambda^{\mathrm{R}}_{\mu})^N\mathbf{P}^{\mathrm{R}}_{\mu})\mathbf{q}=\sum_{\mu=\pm}((\lambda^{\mathrm{F}}_{\mu})^N\mathbf{P}^{\mathrm{F}}_{\mu}) \tag{2.34}$$

and (2.24) follows as a result.

Having observed that the FCM is reproduced in this formulation, we are now at liberty to modify the model by changing the phase rule. We do this simply by changing the contractors $\mathbf{q}^*$ and $\mathbf{q}$. For example, we note that if instead of calculating the propagator corresponding to equation (2.20), we remove the i from the calculation and compute

$$K_{\mathrm{G}}(b,a)=\sum_{R=0,4,\ldots}N(R)(\varepsilon m)^R-\sum_{R=2,6,\ldots}N(R)(\varepsilon m)^R+\left(\sum_{R=1,5,\ldots}N(R)(\varepsilon m)^R-\sum_{R=3,7,\ldots}N(R)(\varepsilon m)^R\right) \tag{2.35}$$

then the corresponding contractors are

$$\mathbf{S}=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}^T \tag{2.36}$$

and

$$\mathbf{S}^*=\begin{pmatrix}1&-1&0&0\\0&0&1&-1\end{pmatrix} \tag{2.37}$$

If we then compute a scalar propagator by taking the trace of the 2×2 contraction, we find

$$K_{\mathrm{G}}(p,t)=\mathrm{Tr}\left[\mathbf{S}^*\left(\lim_{\varepsilon\to 0}\mathbf{T}_4^N\right)\mathbf{S}\right]=2\cos(Et) \tag{2.38}$$

Here the association of phase with path is simpler than in the FCM. In this case there is a phase change of π for each two successive corners in the path, with no phase change for a single corner. We shall call this the KG phase rule, since the resulting propagator $K_G(p, t)$ satisfies the Klein–Gordon equation.

Similarly, we may modify the calculation yet again and for example remove the two minus signs from (2.35), simply adding the sums. It is found that this in fact removes the calculation from the eigenspace corresponding to the two complex eigenvalues $\lambda_{\pm}^{c}$, and the resulting "propagator" is in fact a classical partition function and is *diffusive* in character.

This indicates that the aspect of the FCM that really generates quantum interference is the phase change of π for *each successive pair* of corners in the path. That is, *the KG phase rule is responsible for quantum interference.* (The intermediate phase shift of $\pi/2$ in the FCM which distinguishes left and right is responsible for the particle "spin." Note that the full Feynman phase rule contains the KG phase rule but with left- and right-directed paths distinguished by the factor of i.)

The above observation is highly suggestive. The "classical phase shift" associated with a classical particle confined to the chessboard is precisely π for each successive pair of corners provided those corners have the same chirality (i.e., two successive corners correspond to time reversal). Had the FCM corners had the same chirality, we could have identified the KG phase rule with the classical phase rule, thereby associating quantum interference with a physical law (charge conservation).

Since successive corners in the FCM in fact have opposite chirality, the above association fails and the quantum phase rules still are simply prescriptions with no classical motivation.

However, in the next section we shall modify the microscopic geometry of the paths in the FCM so that successive corners *do* have the same chirality. This will yield paths in which the classical phase rule and the KG phase rule are identical. The task will then be to show that the modified paths combined with the KG/classical phase rule still yield quantum propagators.

3. THE EXTENDED MODEL

In the previous section we constructed quantum propagators by associating phases with certain types of walks on a space-time lattice. The phase rule was implemented through the use of contractors which reduced a 4×4 representation to a 2×2 representation. The 4×4 representation contained the two states of the FCM plus two auxiliary states (2 and 4) which had no direct physical interpretation. The extra states were a convenience which allowed us to reproduce the periodicity of i^R in equation (2.1).

In this section we shall take a different approach. We shall change the geometry of the underlying paths in order that the extra states of the 4×4 formulation have a classical significance.

To this end, consider Figure 3. Here state one corresponds to a particle moving in the $+x$, $+t$ direction. Similarly states two, three, and four correspond respectively to $(-x, -t)$, $(-x, t)$, and $(x, -t)$ directions. If we keep the periodic structure of the FCM, then the sequence $1 \to 3 \to 2 \to 4 \to 1 \ldots$ of the previous models suggests we consider spiral walks as in the figure.

We note that particles executing such walks will in general visit more than one space-time vertex at a given t. This means that in principle a single space-time trajectory could build up a "field" with finite extent in space. Although such a field would be built at the expense of particle number conservation, interpretation of states two and four as antiparticle states would provide a charge conservation principle instead. For example, if we consider states one and three to be electron states, then two and four are positron states and have opposite charge. This means that any single spiral electron trajectory may intersect the line $t = t_0$ many times, but the total charge along the line will be the same for all t_0.

If we interpret states two and four as antiparticle states, then the merging contractors **S*** and **q*** of the previous section may be seen as enforcing

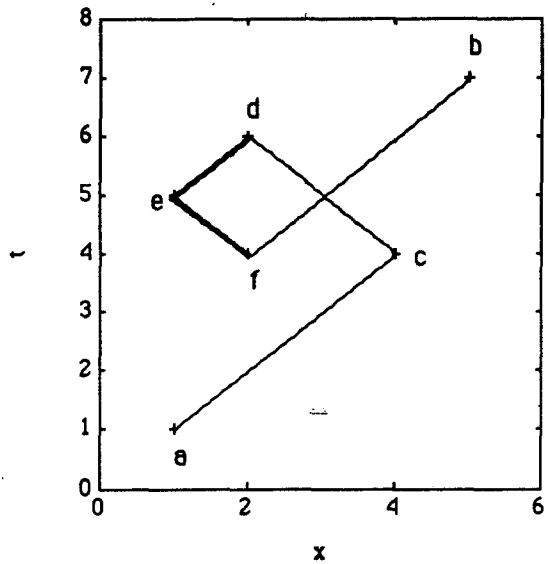

Fig. 3. A spiral path from a to b in the modified model. The particle starts in state 1 at a, switches to state 3 at c, state 2 at d, state 4 at e, and back to state 1 at f. Between d and f the particle has reversed proper time and states 2 and 4 correspond to antiparticle states.

charge conservation, the minus signs subtracting off contributions due to antiparticle states. For example, if T is a 4×4 transfer matrix, then

$$\mathbf{S}^*\mathbf{TS} = \begin{pmatrix} t_{11} - t_{21} & t_{13} - t_{23} \\ t_{31} - t_{41} & t_{33} - t_{43} \end{pmatrix} \tag{3.1}$$

The (1, 2) element of this subtracts transitions from the "particle" state three to the "antiparticle" state two from transitions to the "particle" state one. Similarly, the other elements conserve charge by subtracting contributions from corresponding antiparticle states. This means that the contractors S and S* implement a phase rule corresponding to conservation of charge when applied to the spiral trajectories of Figure 3. Putting this another way, S and S* implement both the KG and the classical phase rules in this system, since here these rules are identical.

Contractors **q*** and **q** also implement conservation of charge; however, they also distinguish states one and three by a phase angle of $\pi/2$.

In order to consider a sum-over-paths propagator with these spiral trajectories, we shall have to keep track of the displacement of the walks in the t direction. In the previous model there was no need to do this, since the walk displacements in t were simply $N\varepsilon$, where N was the number of steps in the walk. With spiral walks this is no longer the case and we shall have to go over to a grand ensemble and let N vary. We did this as an exercise for the FCM model in equation (2.17). The procedure here is similar; we need only produce the relevant transfer matrix. To this end, consider the matrix

$$\mathbf{T}_s = \begin{pmatrix} e^{-i(p+q)\varepsilon} & 0 & 0 & \varepsilon m \\ 0 & e^{i(p+q)\varepsilon} & \varepsilon m & 0 \\ \varepsilon m & 0 & e^{i(p-q)\varepsilon} & 0 \\ 0 & \varepsilon m & 0 & e^{-i(p-q)\varepsilon} \end{pmatrix} \tag{3.2}$$

Here q is the variable conjugate to t, and the diagonal elements of the spiral transfer matrix $\mathbf{T}_s$ increase or decrease the "counting variables" $p\varepsilon$ and $q\varepsilon$ according to the respective displacements in space and time. The off-diagonal elements are the real, positive weights associated with each of the corners in the trajectory.

Our prescription for finding the propagator $\tilde{K}(p, t)$ for this model is similar to the 2×2 case equation (2.17). We have

$$\tilde{K}_{\mathrm{G}}(p, t) = \mathrm{Tr}(\mathbf{S}^*\mathbf{Z}_s\mathbf{S}) \tag{3.3}$$

where for $t=N\varepsilon$

$$\mathbf{Z}_s=\begin{cases}\lim_{\varepsilon\to 0}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d(q\varepsilon)\sum_{n=0}^{\infty}(\mathbf{T}_s)^n e^{iq\varepsilon N}, & t>0\\ 0, & t>0\end{cases} \tag{3.4}$$

In (3.4) the sum on n sums over all path lengths, and the integral over $(q\varepsilon)$ extracts contributions at fixed t. The requirement that $\mathbf{Z}_s$ be zero for $t<0$ corresponds to the calculation of the retarded propagator. Left and right multiplication of $\mathbf{Z}_s$ by $\mathbf{S}^*$ and $\mathbf{S}$, respectively, in (3.3) implements the phase rule of charge conservation, and taking the trace reduces the kernel to a scalar quantity.

Similarly, we might consider forming a 2×2 propagator by using the contractors $\mathbf{q}^*$ and $\mathbf{q}$. That is, we shall also consider

$$\bar{\mathbf{K}}(p,t)=\mathbf{q}^*\mathbf{Z}_s\mathbf{q} \tag{3.5}$$

To calculate (3.4) and (3.5), we proceed formally and write

$$\mathbf{Z}_s=\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} dq\sum_{\mu=1}^{4}\left(\frac{\mathbf{Q}_\mu}{1-\lambda_\mu}\right)e^{iq\varepsilon N}, \qquad t>0 \tag{3.6}$$

where the $\mathbf{Q}_\mu$ are projection operators for $\mathbf{T}_s$. Now equation (3.6) may be evaluated directly, but since the calculation is tedious and the projectors $\mathbf{Q}_\mu$ not particularly enlightening, we postpone the computation in favor of a fairly transparent short-cut.

If we consider calculating the propagator (3.5), one possibility is that we take the contractors inside the sum in (3.4). We shall then be interested in terms of the form $\mathbf{q}^*\mathbf{T}_s^n\mathbf{q}$. We note that

$$\mathbf{q}^*\mathbf{q}=\mathbf{1} \tag{3.7}$$

and to lowest order in ε

$$\mathbf{q}^*\mathbf{T}_s\mathbf{q}=\begin{pmatrix}1-i\varepsilon(p+q) & \varepsilon m\\ \varepsilon m & 1+i\varepsilon(p-q)\end{pmatrix}=\mathbf{T}_F' \tag{3.8}$$

where $\mathbf{T}_F'$ is the transfer matrix (2.16). Similarly, we find that

$$\mathbf{q}^*\,T_s^2\mathbf{q}=(\mathbf{T}_F')^2+\mathbf{O}(\varepsilon^2) \tag{3.9}$$

where $\mathbf{O}(\varepsilon^2)$ is a matrix of terms which are of order ε^2. If we now assume that to lowest order in ε

$$\mathbf{q}^*\mathbf{T}_s^n\mathbf{q}=(\mathbf{T}_F')^n \tag{3.10}$$

then using (3.4), (3.5) becomes

$$\tilde{\mathbf{K}}(p, t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} d(q\varepsilon) \sum_{n=0}^{\infty} (\mathbf{T}_{\mathrm{F}}')^n e^{iq\varepsilon N} \tag{3.11}$$

for $N\varepsilon = t$. We have already evaluated this expression in (2.19) and the result is that $\tilde{\mathbf{K}}(p, t)$ is just the Dirac propagator of (2.15).

Although the above "derivation" is suspect since the assumption (3.10) is not obvious for large n, the procedure does display the relation between the spiral model and the FCM in a particularly simple way.

To see this, note that we have chosen spiral paths in space-time in order to provide a classical motivation for the "quantum" phase rules. In order to do this, we expanded the representation to a 4×4 system, which allowed antiparticle states. This effectively provided us with a classical motivation for the *negative* contributions to the sum over paths. In the 4×4 representation the Feynman phase rule was implemented through contraction to a 2×2 formulation using the contractors $\mathbf{q}^*$ and $\mathbf{q}$. Equation (3.8) states that in the (p, q) representation the contracted spiral transfer matrix is to first order in ε precisely the transfer matrix of the FCM in the same representation. This suggests that *fluctuating chain length and charge conservation in the* 4×4 *formulation yield an effective phase change per unit time equivalent to that of the FCM.*

Similarly, equation (3.3) can be calculated by contracting to a 2×2 representation before summing the series. The result is the Klein–Gordon propagator (2.38). This is particularly interesting because the phase rule in this case is charge conservation alone. There is no extra phase change of $\pi/2$ distinguishing the two directions in space.

Having obtained the propagators (3.3) and (3.5) using heuristic arguments, we now turn to a straightforward evaluation of the 4×4 kernel $\mathbf{Z}_s$ of equation (3.6). The eigenvalues of $\mathbf{T}_s$ are found to be

$$\lambda = 1 \pm i\varepsilon F^{\pm} \tag{3.12}$$

with

$$F^{\pm} = [(q^2 + p^2) \pm (m^4 + 4p^2q^2)^{1/2}]^{1/2} \tag{3.13}$$

We label conjugate pairs as

$$\lambda_{\pm}^{\mathrm{c}} = 1 \pm i\varepsilon F^{-} \tag{3.14}$$

and

$$\lambda_{\pm}^{\mathrm{R}} = 1 \pm i\varepsilon F^{+} \tag{3.15}$$

The eigenvectors corresponding to λ_μ^c for $\mu = \pm$ are

$$\Phi_c^\mu = \frac{i}{2\mu\Delta F^- m^3} \begin{pmatrix} [(F^+)^2 - (d^+)^2](\mu F^- - d^+) \\ (\mu F^- - d^-)m^2 \\ \{[(F^+)^2 - (d^+)^2] + (d^- - \mu F^-)(d^+ - d^-)\}m \\ m^3 \end{pmatrix} \tag{3.16}$$

where

$$d^\pm = (q \pm p) \tag{3.17}$$

and

$$\Delta = [(F^+)^2 - (F^-)^2] \tag{3.18}$$

The eigenvectors corresponding to $\lambda_\pm^R$ can be obtained from (3.16) by interchanging F^+ and F^-. The vectors dual to Φ_c^μ can be found to be

$$\hat{\Phi}_c^\mu = \begin{pmatrix} -im^3 \\ im(\mu F^- + d^+)(\mu F^- + d^+) \\ m^2(\mu F^- + d^+) \\ (\mu F^- + d^-)[(d^+)^2 - (F^-)^2] \end{pmatrix} \tag{3.19}$$

with the vectors $\hat{\Phi}_R^\mu$ being obtained by an interchange of F^+ and F^-. The projection operators can then be formed from (3.16) and (3.19) and the result substituted into equation (3.6). For example, the (1, 1) element of $\mathbf{Z}_s$, corresponding to the first eigenvalue using (3.16) and (3.17) is

$$\begin{aligned} \mathbf{Z}_{s_{11}}^1 &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} d(q\varepsilon) \left(\frac{[(F^+)^2 - (d^+)^2](F^- - d^+)}{2i\varepsilon(\Delta F^-)^2} \right) e^{iq\varepsilon N} \\ &= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} dq \left(\frac{(-2pq + R)[p^2 + q^2 - R - (p+q)]}{4R(p^2 + q^2 - R)} \right) e^{iqt} \end{aligned} \tag{3.20}$$

Here

$$R = (m^4 + 4p^2q^2)^{1/2} \tag{3.21}$$

and we have taken the limit in such a way that $t = N\varepsilon$ is fixed. The superscript on $\mathbf{Z}_s$ indicates that we are considering the first eigenvalue and projector. For $p \ll m$ the integral in (3.20) has poles at $q = \pm E$ and branch points at $q = \pm im^2/2p$. As in the evaluation of (2.19), we move the poles slightly into

the upper half-plane and complete the contour using a large semicircle in the upper half-plane. This time, however, there is a branch cut along the imaginary q axis extending from the point $q=im^2/2p$ to $q=+i\infty$, and the semicircular arc must be modified to exclude the cut. This gives a contribution to the integral which decays as $e^{-m^2t/2p}$. This may be neglected for time scales much greater than the Compton wavelength $t \gg 1/m$. The contribution from the two poles then yields

$$\mathbf{Z}^1_{s_1} = \tfrac{1}{8}[(1+p/E)\, e^{-iEt} + (1-p/E)\, e^{iEt}], \qquad t>0 \tag{3.22}$$

A similar calculation for the (2, 1) element of this projector shows that

$$\mathbf{Z}^1_{s21} = -\mathbf{Z}^1_{s11} \tag{3.23}$$

The same calculation for the conjugate eigenvalue $\lambda = 1 - i\varepsilon F^-$ yields identical contributions. The integrals for the two remaining eigenvalues $\lambda^{\mathrm{R}}_{\pm}$ are similar; however, as in the 4×4 version of the FCM, both sets of contractors $\mathbf{q}^*$ and $\mathbf{q}$, and $\mathbf{S}^*$ and $\mathbf{S}$ eliminate the contributions from these eigenvalues. As a result, the (1, 1) element of the 2×2 propagator is, from (3.5)

$$\begin{aligned} \tilde{\mathbf{K}}_{11}(p, t) &= \sum_{\mu=1}^{2} (\mathbf{Z}^{\mu}_{s11} - \mathbf{Z}^{\mu}_{s21}), \qquad t>0 \\ &= \tfrac{1}{2}[(1+p/E)\, e^{-iEt} + (1-p/E)\, e^{iEt}] \end{aligned} \tag{3.24}$$

This is just the (1, 1) element of $\mathbf{K}(p, t)$ from equation (2.15).

In a similar way, all four elements of $\tilde{\mathbf{K}}(p, t)$ can be evaluated, and the result is that $\tilde{\mathbf{K}}(p, t)$ is the Dirac propagator of equation (2.15). Implementing the contraction using the matrices $\mathbf{S}^*$ and $\mathbf{S}$ also yields similar results and we find that $\tilde{\mathbf{K}}_{\mathrm{G}}(p, t) = \mathbf{K}_{\mathrm{G}}(p, t)$.

4. CONCLUSIONS

In Section 2 we reviewed two formulations of the FCM. The first formulation, due to Gersch (1981) and Jacobson and Schulman (1984), calculated the sum over paths using a 2×2 transfer matrix which implemented the FCM phase rule by associating a phase angle of $\pi/2$ with each corner in the Feynman paths. In this formulation the transfer matrix had imaginary off-diagonal elements and the calculation of the propagator corresponded to the calculation of a partition function with imaginary Boltzman weights.

The second formulation, following Ord (1992), expanded the representation to a 4×4 system in which the transfer matrix corresponded to a calculation of a classical partition function with real, positive weights. In

this formulation the FCM phase rule was implemented after the calculation of the partition functions through contraction to a 2×2 representation. The contraction process removed the eigenvalue which dominated the classical partition function calculation, and selected subdominant behavior which reflected the periodic structure of the transfer matrix. This showed that the calculation of the quantum propagator was in a sense "embedded" in the calculation of a classical partition function for a four-state process, but the phase rule for extracting this propagator lacked any classical motivation.

In Section 3 we used the 4×4 formulation of the preceding section to consider a sum over paths where the paths were all constrained to be spirals in space-time. The reason for considering such trajectories was to use the time-reversed portions of the paths to provide a classical motivation for the phase rule of the FCM. This strategy worked very well, particularly in the case of the Klein–Gordon propagator. In that case all that was needed to extract the propagator from the subdominant behavior of the classical partition function was the single rule of charge conservation.

It is interesting to note that this calculation suggests that the relationship between quantum mechanics and special relativity may be closer than previously suspected. The "derivation" of the Klein–Gordon kernel from the sum over spiral paths used only motivation from special relativity. Although this derivation is not appropriate at high energies, it is valid in the nonrelativistic regime and as such provides a physical mechanism for quantum interference in this regime. Put another way, for a nonrelativistic free particle in one dimension there is now a microscopic basis (i.e., charge conservation on time scales of the order of the Compton wavelength) for the usual, purely formal method of quantization involving analytic continuation (e.g., $t \to it$). This physical basis has its roots in special relativity.

So far we have made no attempt to interpret "observation" in this system. However, it is interesting to speculate that the above model may provide an example of a system where the "statistical interpretation" of quantum mechanics has some physical basis. Since the spiral paths in space-time effectively "traverse" many paths of the conventional Feynman formulation, there is some hope that the ensemble averages of the conventional formulation may be closely related to "time" averages of single spiral trajectories.

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